

Moment matching for bilinear systems with nice selections

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Abstract: The paper develops a method for model reduction of bilinear control systems. It leans upon the observation that the input-output map of a bilinear system has a particularly simple Fliess series expansion. Subsequently, a model reduction algorithm is formulated such that the coefficients of Fliess series expansion for the original and reduced systems match up to certain predefined sets - nice selections. Algorithms for computing matrix representations of unobservability and reachability spaces complying with a nice selection are provided. Subsequently, they are used for calculating a partial realization of a given input-output map.

1. INTRODUCTION

The paper deals with model reduction of bilinear systems. Bilinear systems appear in many applications, and there is an extensive literature on their analysis and control Isidori (1989); Elliott (2009). However, bilinear models occurring in practice are often of such large dimensions that application of existing control synthesis and analysis methods is not feasible. For this reason, it is of interest to investigate model reduction algorithms for bilinear systems.

Problem Formulation

In the sequel we give a brief description of the system to be studied, for details we refer to Section 2. Consider a bilinear system of the form

$$\dot{x} = A_0x + \sum_{i=1}^m (A_i x)u_i, \quad y = Cx,$$

with $A_0, \dots, A_m \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{p \times n}$, and where x is the state trajectory, $u = (u_1, \dots, u_m)$ is the input trajectory and y is the output trajectory. For a fixed initial state x_0 , consider the input-output map \mathcal{Y}_{x_0} of the system induced by the initial state x_0 . Specifically, $\mathcal{Y}_{x_0}(u, t)$ is the output $y(t)$ of the system at time t , corresponding to the input u and initial state $x(0) = x_0$.

In this paper, we propose an algorithm for finding a bilinear system of reduced order:

$$\dot{\bar{x}} = \bar{A}_0\bar{x} + \sum_{i=1}^m (\bar{A}_i\bar{x})u_i, \quad \bar{y} = \bar{C}\bar{x}$$

with $\bar{A}_0, \dots, \bar{A}_m \in \mathbb{R}^{\bar{n} \times \bar{n}}$ and $\bar{C} \in \mathbb{R}^{p \times \bar{n}}$, $\bar{n} < n$, and an initial state \bar{x}_0 such that the input-output map $\bar{\mathcal{Y}}_{\bar{x}_0}$ of the reduced order system induced by the initial state \bar{x}_0 is close to \mathcal{Y}_{x_0} . More precisely, the algorithm guarantees that:

- (i) $\mathcal{Y}_{x_0}(u, t) = \bar{\mathcal{Y}}_{\bar{x}_0}(u, t)$ for all $u \in \mathcal{U}$ and $t \in [0, T]$, where \mathcal{U} is a priori chosen subset of input signals, and
- (ii) The Euclidian 2-norm $\|\mathcal{Y}_{x_0}(u, t) - \bar{\mathcal{Y}}_{\bar{x}_0}(u, t)\|$ is small for small t and u .

In the paper, we will provide a concrete error bound for $\|\mathcal{Y}_{x_0}(u, t) - \bar{\mathcal{Y}}_{\bar{x}_0}(u, t)\|$ as a function of the magnitude of t and u . Moreover, we will discuss how to choose the class of inputs \mathcal{U} in item (i) above. The algorithm resembles that in Bastug et al. (2016) developed for the model reduction of linear switched systems. In a nutshell, it relies on *matching* a number of coefficients of the Fliess series (or functional) expansion of \mathcal{Y}_{x_0} and $\bar{\mathcal{Y}}_{\bar{x}_0}$, i.e. $\bar{\Sigma}$ is constructed in such a way that a certain coefficients of the Fliess-series expansion of \mathcal{Y}_{x_0} and $\bar{\mathcal{Y}}_{\bar{x}_0}$ are the same. The set of indices of these coefficients is called a *nice selection*, and it is a subset of sequences of elements from $\{0, 1, \dots, m\}$. Intuitively, various choices of nice selections correspond to choosing those Fliess-series coefficients which contribute the most to responses to specific input trajectories. In particular, for certain inputs, the coefficients specified by nice selections completely determine the output response of the system. In short, we use the nice selection to keep the dimension of the reduced system low by pinpointing our attention on specific input trajectories.

Prior work To the best of our knowledge, the contribution of the paper is new. Model reduction of bilinear systems is an established topic, without claiming completeness, we mention Bai and Skoogh (2006); Zhang and Lam (2002); Wang and Jiang (2012); Xu et al. (2015); Breiten and Damm (2010); Benner and Breiten (2015); Flagg and Gugercin (2015); Feng and Benner (2007); Lin et al. (2007); Flagg (2012). In particular, moment matching methods for bilinear systems were proposed in Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007); Flagg (2012); Breiten and Damm (2010); Benner and Breiten (2015); Flagg and Gugercin (2015); Wang and Jiang (2012) and for general non-linear systems in Astolfi (2010). The current paper represents another version of moment matching for bilinear systems. In Flagg (2012); Breiten and Damm (2010); Benner and Breiten (2015) the concept of moment matching at some frequencies $\sigma_1, \dots, \sigma_k$ was defined. The papers Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007) correspond to moment matching at zero frequency ($\sigma = 0$). The cur-

rent paper proposes the notion of γ -partial realization for some nice selection γ , which is a generalization of moment matching at ∞ proposed in Breiten and Damm (2010). The precise relationship is explained in Remark 2. Since the notion of γ -partial realization is more general than moment matching at ∞ , the algorithm of this paper is rather different from those of Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007); Flagg (2012); Breiten and Damm (2010); Benner and Breiten (2015); Flagg and Gugercin (2015); Wang and Jiang (2012). In particular, we cannot reduce the problem to computing classical Krylov-subspaces, and hence we cannot use the corresponding rich mathematical structure. As a result, the numerical issues of the proposed algorithm are much less clear than those of the algorithms cited above. Moreover, the interpretation of the model reduction procedure as an approximation in H_2 does not carry over to the framework of the current paper either. The relationship between the proposed method and moment matching at other frequencies remains the topic of future research. The same holds for the relative merits of Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007); Flagg (2012); Breiten and Damm (2010); Benner and Breiten (2015); Flagg and Gugercin (2015); Wang and Jiang (2012). At this stage, it is not clear in which situations the approach proposed in this paper is more useful than the algorithms from Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007); Flagg (2012); Breiten and Damm (2010); Benner and Breiten (2015); Flagg and Gugercin (2015); Wang and Jiang (2012). An advantage of the proposed method is that it allows to choose the order of the reduced-order system a-priori, by choosing the number of elements of a nice selection, see Remark 3 of Section 4. In addition, to the best of our knowledge, this paper is the first to present a characterization of those inputs, for which the corresponding input-output behavior is preserved by moment matching. In this sense, the paper follows the spirit of Astolfi (2010), although the technical details of the definition are quite different. Furthermore, the paper provides some error bounds, which were absent from the existing literature.

Contents of the paper This work is organized as follows. In Section 2, we introduce the notation used throughout the paper; subsequently, we recall the highlights of the realization theory for bilinear systems. In Section 3, we introduce the concepts of a column nice selection and a row nice selection, which we use for formulating a convenient partial realization of bilinear systems. Subsequently, we characterize the level at which a bilinear system - a partial realization of f - approximates f . Lastly, in Section 4, we equip the reader with algorithms for computing partial realizations for nice selections.

2. PRELIMINARIES

2.1 Notation

Denote by \mathbb{N} the set of natural numbers including 0, and by \mathbb{R}_+ the set $[0, +\infty)$ of nonnegative real numbers. The symbol $\|\cdot\|$ will denote the Euclidean 2-norm when applied to vectors and induced 2-norm when applied to matrices. If F is a function between function spaces we write $F[u]$ in place of $F(u)$ to distinguish the arguments e.g., $F[u](t)$ indicate that $F[u]$ is function of t .

In addition, $AC(\mathbb{R}_+, \mathbb{R}^n)$ denotes the set of *absolutely continuous maps*, and $L_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ the set of *Lebesgue measurable maps* which are integrable on any compact interval. The time derivative of $x \in AC(\mathbb{R}_+, \mathbb{R}^n)$ is denoted \dot{x} where it is implicitly understood that this notation indicates almost everywhere differentiable.

Let Q denote the set $\{0, 1, \dots, m\}$, and Q^* the set of finite sequences of elements of Q together with the empty sequence ϵ . Let $w = q_1 q_2 \dots q_k \in Q^*$ with $q_1, \dots, q_k \in Q$, $k > 0$ and $A_{q_i} \in \mathbb{R}^{n \times n}$, $i = 1, \dots, k$. Then the matrix A_w is defined as

$$A_w = A_{q_k} A_{q_{k-1}} \dots A_{q_1}.$$

By convention, if $w = \epsilon$, then A_ϵ is the identity matrix. A bilinear system is a (control) system of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^m (A_i x(t)) u_i(t), \quad x(0) = x_0 \quad (1a)$$

$$y(t) = C x(t) \quad (1b)$$

with $A_i \in \mathbb{R}^{n \times n}$, $i \in Q$, $C \in \mathbb{R}^{p \times n}$, and where $u = (u_1, \dots, u_m) \in L_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ is the input trajectory, $x \in AC(\mathbb{R}_+, \mathbb{R}^n)$ is the state trajectory, and $y \in AC(\mathbb{R}_+, \mathbb{R}^p)$ is the output trajectory.

The notation $\Sigma = (p, m, n, \{A_i\}_{i \in Q}, C, x_0)$ or simply Σ , is used as short-hand representations for a bilinear system of the form (1). The number n is the *dimension (or order)* of Σ and is sometimes denoted by $\dim \Sigma$.

The *input-to-state* map $X_{\Sigma, x}$ and *input-to-output* map $Y_{\Sigma, z}$ of Σ are the maps

$$X_{\Sigma, z} : L_{loc}(\mathbb{R}_+, \mathbb{R}^m) \rightarrow AC(\mathbb{R}_+, \mathbb{R}^n); \quad u \mapsto X_{\Sigma, z}[u],$$

$$Y_{\Sigma, z} : L_{loc}(\mathbb{R}_+, \mathbb{R}^m) \rightarrow AC(\mathbb{R}_+, \mathbb{R}^p); \quad u \mapsto Y_{\Sigma, z}[u]$$

defined by letting $t \mapsto X_{\Sigma, z}[u](t)$ be the solution to the Cauchy problem (1a) with $x_0 = z$, and letting $Y_{\Sigma, z}[u](t) = C X_{\Sigma, z}[u](t)$ as in (1b).

2.2 Realization of bilinear systems

Below, we recall elements of realization theory for bilinear systems Rugh (1981). The bilinear system Σ is a *realization* of a function

$$f : L_{loc}(\mathbb{R}_+, \mathbb{R}^m) \rightarrow AC(\mathbb{R}_+, \mathbb{R}^p), \quad (2)$$

if $f = Y_{\Sigma, x_0}$. The system Σ is a *minimal realization* of f , if it has the smallest state-space dimension among all bilinear systems which are realizations of f . Bilinear system Σ is *observable*, if for any two states $x_1, x_2 \in \mathbb{R}^n$, $Y_{\Sigma, x_1} = Y_{\Sigma, x_2}$ implies $x_1 = x_2$. We say that Σ is *span-reachable*, if $\text{Span}\{X_{\Sigma, x_0}[u](t) \mid t \geq 0, u \in L_{loc}(\mathbb{R}_+, \mathbb{R}^m)\} = \mathbb{R}^n$. It is well-known Isidori (1989) that Σ is a minimal realization of f , if and only if Σ is a realization of f , and it is span-reachable and observable. Span-reachability and observability have algebraic characterizations. Specifically, Σ is observable, if and only if

$$\bigcap_{w \in Q^*} \ker C A_w = \{0\};$$

and Σ is span-reachable if and only if

$$\text{Span}\{A_w x_0 \mid w \in Q^*\} = \mathbb{R}^n.$$

In the sequel, we use the notion of generating series for bilinear systems Isidori (1989); Rugh (1981); Gray and Wang (2002).

Definition 1. A *generating series* (over Q) is a function $c : Q^* \rightarrow \mathbb{R}^p$ such that there exist $K, R > 0$ which satisfy

$$\forall w \in Q^* : \|c(w)\| \leq KR^{|w|}, \quad (3)$$

where $|w|$ is the length of the sequence w (number of elements from Q in w).

For each $u \in L_{loc}(\mathbb{R}_+, \mathbb{R}^m)$, $w \in Q^*$ and $t \geq 0$, define the iterated integral $V_w[u](t)$ as follows:

- (1) $V_\epsilon[u](t) = 1$,
- (2) for $w = q_1 \cdots q_k \in Q^*$, $q_1, \dots, q_k \in Q$, and $k > 1$

$$V_w[u](t) = \int_0^t u_{q_k}(\tau) V_{q_1 \cdots q_{k-1}}[u](\tau) d\tau,$$

where $u_0(t) = 1$.

For $q \in Q$ we note that $V_q[u](t) = \int_0^t u_q(\tau) d\tau$ since $q = q\epsilon$. We define the function $F_c : L_{loc}(\mathbb{R}_+, \mathbb{R}^m) \rightarrow L_{loc}(\mathbb{R}_+, \mathbb{R}^p)$ generated by a *generating series* c by

$$F_c[u](t) = \sum_{w \in Q^*} c(w) V_w[u](t). \quad (4)$$

The map F_c is called a *Fliess operator* and the right-hand side of (4) a *Fliess series*. The Fliess operator F_c is well-defined, as the growth condition (3) is sufficient for absolute convergence of the series in (4), by Theorem 3.1, Gray and Wang (2002).

Following Isidori (1989), f has a realization by a bilinear system, only if $f = F_{c_f}$ for some generating series c_f . Moreover, a bilinear system Σ of the form (1) is a realization of f , if and only if $\forall w \in Q^* : c_f(w) = CA_w x_0$.

3. NICE SELECTION FOR BILINEAR SYSTEMS

At the outset, we define a concept of nice selections.

Definition 1. (Nice selections). A subset α of Q^* is called a *column nice selection* of a bilinear system Σ of the form (1), if α has the following property, which we refer to as *prefix closure*: if $wq \in \alpha$ for some $q \in Q$, $w \in Q^*$, then $w \in \alpha$. A subset β of Q^* is called a *row nice selection* of Σ if β has the following property, which we refer to as *suffix closure*: if $qw \in \beta$ for some $q \in Q$, $w \in Q^*$, then $w \in \beta$.

The phrase "nice selection" will be used when it is irrelevant whether a nice row or column selection is used.

Based on a nice selection, we next introduce the notion of a γ -partial realizations of an input-output map by Σ .

Definition 2. (γ -partial realization). Let γ be a nice selection of a bilinear system Σ of the form (1) and let f be an input-output map of the form (2), where it is assumed that $f = F_{c_f}$ for a generating series c_f . The bilinear system Σ is then called a γ -partial realization of f , if

$$\forall w \in \gamma : c_f(w) = CA_w x_0.$$

Definition 2 states that Σ is an γ -partial realization of f , if the generating series of Y_{Σ, x_0} and that of f coincide on the set γ .

Remark 1. (Classical partial realization). Let γ be the set of sequences of length at most N , $\gamma = \{v \in Q^* \mid |v| \leq N\}$. In this case we note that γ -partial realization reduces to N -partial realization as defined in Isidori (1973, 1989). In particular, if $N \geq \max\{n, n'\}$, where $n = \dim \Sigma$ and n' is

the dimension of a minimal bilinear realization of f and Σ is a γ -partial realization of f , then Σ is a realization of f .

Remark 2. (Relationship with moment matching). As it was mentioned in the introduction, moment matching for bilinear systems was investigated in Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007); Flagg (2012). Below, we will explain in detail the relationship between the existing definition for moment matching and Definition 2 for γ -partial realization. Note that in the cited papers, the systems of the form $\dot{z} = Az + \sum_{i=1}^m N_i z u_i + Bu$, $y = Hz$, $z(0) = 0$ were studied. By defining $x = (z^T, 1)^T$,

$$A_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} N_i & B_i \\ 0 & 0 \end{bmatrix}, \quad i = 1, \dots, m, \quad C = [H \quad 0],$$

where B_i is the i th column of B ; it is clear that x and y satisfy (1). Hence, the system class considered in those paper can be embedded into the system class considered in this paper. We will consider $m = 1$ in order to avoid excessive notation. Consider a bilinear system $\Sigma = (p, 1, n, \{A_0, A_1\}, C, x_0)$ and assume that Σ is a realization of f . In Breiten and Damm (2010); Benner and Breiten (2015); Flagg and Gugercin (2015); Wang and Jiang (2012) the moments $m_\Sigma(l_1, \dots, l_k)$, $0 < l_1, \dots, l_k \in \mathbb{N}$ of Σ at certain frequencies $\sigma_1, \dots, \sigma_k$ were defined as follows: $m_\Sigma(l_1, \dots, l_k) = C(\sigma_1 I - A_0)^{-l_1} A_1 (\sigma_2 I - A_0)^{-l_2} \cdots A_1 (\sigma_k I - A_0)^{-l_k} x_0$ for all if $\sigma_1, \dots, \sigma_k \in \mathbb{C}$ and $m_\Sigma(l_1, \dots, l_k) = c_f(0^{l_1-1} 10^{l_2-1} \cdots 10^{l_k-1}) = CA_0^{l_1-1} A_1 \cdots A_1 A_0^{l_k-1} x_0$ if $\sigma_1 = \cdots = \sigma_k = \infty$. In the cited literature, a system $\bar{\Sigma} = (p, 1, n, \{\bar{A}_0, \bar{A}_1\}, \bar{C}, \bar{x}_0)$ was said to match the moments of Σ for $l_1, \dots, l_j \in \{0, \dots, N\}$, $j = 1, \dots, k$, $N \in \mathbb{N}$ at frequencies $\sigma_1, \dots, \sigma_k$, if $m_\Sigma(l_1, \dots, l_j) = m_{\bar{\Sigma}}(l_1, \dots, l_j)$ for all $l_1, \dots, l_j \in \{1, \dots, N\}$, $j = 1, \dots, k$. Note that in Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007); Flagg (2012) only moments for $\sigma_1 = \cdots = \sigma_k = 0$ were considered. Moment matching at ∞ can be expressed in our framework as follows: $\bar{\Sigma}$ matches the moments of Σ for $l_1, \dots, l_j \in \{1, \dots, N\}$, $j = 1, \dots, k$ at the frequency $\sigma_1 = \cdots = \sigma_k = \infty$, if and only if $\bar{\Sigma}$ is a γ -partial realization of Y_{Σ, x_0} , where $\gamma = \{0^{l_1} 1 \cdots 0^{l_{j-1}} 10^{l_j} \mid l_1, \dots, l_j \in \{0, \dots, N-1\}, j = 1, \dots, k\}$. Note that γ is both prefix and suffix closed, i.e., it qualifies both for nice row and nice column selection. For other frequencies, the relationship is less obvious, and it remains a topic of future research.

There are two descriptions of the fact that Σ is an γ -partial realization of f . The first is that the input-output map Y_{Σ, x_0} of Σ is an approximation of f in the sense that for all inputs u , the outputs $Y_{\Sigma, x_0}(u)$ and $f(u)$ are close to each other in a suitable metric. The other interpretation is that for some inputs u , $Y_{\Sigma, x_0}(u)$ equals $f(u)$. Below, we characterize both cases.

Let γ be a nice selection of a bilinear system Σ of the form (1) and for any $T \geq 0$, define

$$\mathcal{U}_{\gamma, T} = \{u \in L_{loc}(\mathbb{R}_+, \mathbb{R}^m) \mid \forall v \in Q^*, v \notin \gamma, t \in [0, T] : V_v[u](t) = 0\}.$$

For $\alpha \in \mathbb{N}$ and $q \in Q$ let q^α denote the sequence $qq \cdots q$ obtained by repeating q α -times. If $\alpha = 0$, let q^0 be the empty word. Define the set

$$L_\gamma = \{v \in Q^* \mid v = q_1 \cdots q_k, \quad q_1, \dots, q_k \in Q, k > 0 \\ \text{such that } \forall v_i \in \{0, q_i\}^*, i = 1, \dots, k, v_1 v_2 \cdots v_k \in \gamma\},$$

and recall that $\{0, q_i\}^*$ denotes the set of all sequences of q_i and 0. In particular, $v \in \{0, q_i\}^*$ if and only if v contains only the symbols 0 and q_i , or, in other words, $v = 0^{l_1} q_i^{l_2} \dots q_i^{l_{r-1}} 0^{l_r}$ for some $l_1, \dots, l_r \in \mathbb{N}$, $r > 0$. We will call L_γ the set of sequences consistent with γ .

Now for any $q \in Q$, let e_q denotes the q th standard basis vector of \mathbb{R}^m , if $q \in \{1, \dots, m\}$, and $e_0 = 0$. We will say that $u \in L_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ is consistent with the nice selection γ on an interval $[0, T]$, if there exist $q_1, \dots, q_k \in Q$, reals $0 < t_1 < \dots < t_{k-1} < t_k = T$ and scalar valued functions $u_i \in L_{loc}([t_i, t_{i+1}], \mathbb{R})$, such that $u(s) = u_i(s)e_{q_i}$, $s \in [t_i, t_{i+1})$ for all $i = 1, \dots, k$, $u(T) = e_{q_k}$ and $q_1 \dots q_k \in L_\gamma$. That is, if u is consistent with γ on $[0, T]$, then $[0, T]$ can be divided into a finite number of intervals, and on each interval, at most one component of u is not zero.

We are now ready to state the main result relating values of Y_{Σ, x_0} and f .

Theorem 1. (Preserving input-output behavior). Let γ be a nice selection, of a bilinear system Σ , containing all zero sequences; $\{0\}^* \subseteq \gamma$.

(A) If Σ is an γ -partial realization of f , then

$$\forall u \in \mathcal{U}_{\gamma, T}, t \in [0, T] : Y_{\Sigma, x_0}(u)(t) = f(u)(t) \quad (5)$$

(B) Any u consistent with γ on $[0, T]$ belongs to $\mathcal{U}_{\gamma, T}$, and $0 \in \mathcal{U}_{\gamma, T}$.

The theorem above claims that if Σ is a γ -partial realization of f , then f and the input-output map of Σ coincide on the set of inputs which are mapped to the zero map by maps $u \mapsto V_w[u]$ indexed by words not in the nice selection γ . Moreover, the set of such inputs includes piecewise-constant inputs of switching type. Theorem 1 mirrors the results of Bastug et al. (2016).

Proof. Part (A) From $u \in \mathcal{U}_{\beta, T}$ it follows that $V_v[u](t) = 0$ for all $v \in Q^*$, $v \notin \gamma$, $t \in [0, T]$, and hence

$$\sum_{v \notin \gamma, v \in Q^*} c(v)V_v[u](t) = 0 = \sum_{v \notin \gamma, v \in Q^*} c_f(v)V_v[u](t),$$

$$\text{and } Y_{\Sigma, x_0}(u)(t) = \sum_{v \in Q^*} c(v)V_v[u](t) = \sum_{v \in \gamma} c(v)V_v[u](t) = \sum_{v \in \gamma} c_f(v)V_v[u](t) = \sum_{v \in Q^*} c_f(v)V_v[u](t) = f(u)(t).$$

Part (B) If u is consistent with γ , then

$$u = \mathbf{u}_1 \#_{\tau_1} \mathbf{u}_2 \#_{\tau_2} \dots \#_{\tau_k} \mathbf{u}_k,$$

where for $i = 1, \dots, k$

$$\tau_1 = t_1, \quad \tau_i = t_i - t_{i-1}, \quad \mathbf{u}(t) = u_i(t)e_{q_i},$$

and where for any two functions f_1, f_2 ,

$$(f_1 \#_{\tau} f_2)(s) = \begin{cases} f_1(s) & s \in [0, \tau) \\ f_2(s - \tau) & s \in [\tau, +\infty) \end{cases}$$

Using (Wang and Sontag, 1992, eq. (11)) repeatedly, it then follows that for any $v \in Q^*$ and for any $t \in [t_{j-1}, t_j)$, $j = 1, \dots, k$,

$$V_v[u](t) = \sum_{\substack{v_1, \dots, v_j \in Q^*, \\ v = v_1 v_2 \dots v_j}} V_{v_1}[\mathbf{u}_1](\tau_1) \dots V_{v_j}[\mathbf{u}_j](\tau_j - t), \quad (6)$$

Note that $\mathbf{u}_i(s) = u_i(s)e_{q_i}$ for all $s \in [0, \tau_i]$ and hence $V_w[\mathbf{u}_i](s) \neq 0 \implies w \in \{0, q_i\}^*$ for all $i = 1, \dots, k$. Hence, from this and (6) it follows that $V_v[u](t) \neq 0$ implies that $v = v_1 \dots v_j$ such that $v_i \in \{0, q_i\}^*$, $i = 1, \dots, j$. If $j < k$, we can take $v_{j+1} = \dots = v_k = \epsilon$, and then $v_i \in \{0, q_i\}^*$,

$i = 1, \dots, k$, $v = v_1 v_2 \dots v_k$. But by the assumption that u is consistent with γ on $[0, T]$, it follows that $q_1 \dots q_k \in L_\gamma$, which implies that $v = v_1 \dots v_k \in \gamma$. That is, if $V_v[u](t) \neq 0$ for some $t \in [0, T]$, then $v \in \gamma$. Hence, for any $v \notin \gamma$, $V_v[u](t) = 0$ for all $t \in [0, T]$. The latter implies that $u \in \mathcal{U}_{\gamma, T}$.

Theorem 1 provides the set of inputs $\mathcal{U}_{\gamma, T}$ such that the γ -partial realisation and the original input-output map coincides. It is desirable that the set $\mathcal{U}_{\gamma, T}$ is as big as possible; however, the price is a high dimension of the reduced system. Therefore, we use a nice selection to keep the dimension of the reduced system low by pinpointing the attention on specific input trajectories - switching inputs of form $u(s) = u_i(s)e_{q_i}$ for s in some time interval $[t_i, t_{i+1})$.

Example 1. Let us take $m = 2$ and let γ to be the set of all sequences containing only 0's and 1's. The set $\mathcal{U}_{\gamma, T}$ will contain all those input signals whose second component is zero on $[0, T]$, i.e., $u = (u_1, u_2) \in \mathcal{U}_{\gamma, T} \iff \forall t \in [0, T] : u_2(t) = 0$.

Let us now take $\gamma = \{v_1 v_2 \mid v_1 \in \{0, 1\}^*, v_2 \in \{0, 2\}^*\}$.

Then for any $T, t_1 > 0$, the input $u(t) = \begin{cases} (1, 0)^T & t < t_1 \\ (0, 1)^T & t_1 \leq t \end{cases}$

will belong to $\mathcal{U}_{\gamma, T}$, but $u(t) = \begin{cases} (0, 1)^T & t < t_1 \\ (1, 0)^T & t_1 \leq t \end{cases}$ will not belong to $\mathcal{U}_{\gamma, T}$, since $V_{21}[u](t) = \int_0^T u_1(\tau) \int_0^\tau u_2(\tau_1) d\tau_1 d\tau = \int_{t_1}^T \int_0^{t_1} d\tau_1 d\tau = t_1(T - t_1) \neq 0$, but the sequence $21 \notin \gamma$.

Let us now investigate the behavior of γ -partial realization of f for all bounded inputs. We formulate the following simple but useful observation, where for any $\gamma \subseteq Q^*$ containing the empty sequence ϵ , we introduce the notation

$$N_\gamma = \max\{N \in \mathbb{N} \mid \{v \in Q^* \mid |v| \leq N\} \subseteq \gamma\}, \quad (7)$$

which in "worst" case is zero.

Lemma 1. Let c_f be the generating series of f and c be the generating series of Y_{Σ, x_0} . Then there exist real numbers $K_\gamma, M_\gamma > 0$ such that

$$\forall w \in Q^* \setminus \gamma : \|c_f(w) - c(w)\| \leq K_\gamma M_\gamma^{|w| - N_\gamma}. \quad (8)$$

Proof. Recall that $c(w) = CA_w x_0$. Since c_f is a generating series, there exists $K_f > 0, M_f > 0$ such that for all $w \in Q^* : \|c_f(w)\| < K_f M_f^{|w|}$. Leaning on the two observations above, we define $M_A = \max\{\|A_0\|, \dots, \|A_m\|\}$, $K'_\gamma = \max\{\|C\| M_A^{N_\gamma} \|x_0\|, K_f M_f^{N_\gamma}\}$, $M_\gamma = \max\{M_f, M_A\}$, and $K_\gamma = 2K'_\gamma$.

Lemma 1 is an ingredient in the proof of Theorem 2, which provide an error bound between the input output map f and a corresponding γ -partial realization, on the input trajectories not belonging to $\mathcal{U}_{\gamma, T}$.

Theorem 2. Let γ be a nice selection and suppose that $u \in L_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ and

$$\sup_{t \in [0, T]} \|u(t)\| < R,$$

and Σ is a γ -partial realization of f . Then there exist reals $K_\gamma, M_\gamma > 0$ such that

$$\forall t \in [0, T] : \|Y_{\Sigma, x_0}(u)(t) - f(u)(t)\| \leq K_\gamma ((m+1)^2 \max\{R, t\})^{N_\gamma} e^{M_\gamma (m+1)^2 \max\{R, t\}} \quad (9)$$

The theorem above complies with the intuition that for sufficiently small time and small inputs, i.e., for $(m+1)^2 \max\{R, t\} < 1$, the larger N_γ the smaller is the difference between f and the input-output map of Σ .

Proof. By Lemma 2.1 in Gray and Wang (2002), if $K = \max\{R, t\}$ then for any $w = q_1, \dots, q_k \in Q^*$,

$$\|V_w[u](t)\| \leq K^k \frac{1}{r_0! \dots r_m!} \quad (10)$$

where $r_i \equiv r_i(w)$ is the number of occurrences of the integer $i \in Q$ in the sequence w . We will use a notation $r(w) = (r_0(w), \dots, r_m(w))$. To prove (9), we use (8) and the following observation. By the binomial expansion, we have $(m+1)^k = ((1 + \dots + 1) + 1)^k = \sum_{|\bar{r}|=k} \frac{k!}{\bar{r}_0! \dots \bar{r}_m!}$ where we have used the notation $|\bar{r}| \equiv \bar{r}_0 + \dots + \bar{r}_m$.

As a consequence,

$$\begin{aligned} \|Y_{\Sigma, x_0}(u)(t) - f(u)(t)\| &\leq \sum_{w \in Q^* \setminus \gamma} \|c_f(w) - c(w)\| \|V_w[u](t)\| \\ &\leq \sum_{w \in Q^* \setminus \gamma} K_\gamma M_\gamma^{|w| - N_\gamma} \frac{K^{|w|}}{r_0(w)! \dots r_m(w)!} \end{aligned}$$

for N_γ defined in (7). We continue

$$\begin{aligned} &\sum_{w \in Q^* \setminus \gamma} K_\gamma (M_\gamma)^{|w| - N_\gamma} \frac{K^{|w|}}{r_0(w)! \dots r_m(w)!} \\ &\leq \sum_{w \in Q^*, |w| \geq N_\gamma} K_\gamma (M_\gamma)^{|w| - N_\gamma} \frac{K^{|w|}}{|w|!} \frac{|w|!}{r_0(w)! \dots r_m(w)!} \\ &= \sum_{k=N_\gamma}^{\infty} K_\gamma (M_\gamma)^{k - N_\gamma} \frac{K^k}{k!} \sum_{\substack{|\bar{r}|=k \\ \bar{r} = (\bar{r}_0, \dots, \bar{r}_m) \\ r(w) = \bar{r}}} \sum_{w \in Q^*} \frac{k!}{\bar{r}_0! \dots \bar{r}_m!} \\ &\leq \sum_{k=N_\gamma}^{\infty} K_\gamma (M_\gamma)^{k - N_\gamma} \frac{K^k}{k!} \left(\sum_{\bar{r}_0 + \dots + \bar{r}_m = k} \frac{k!}{\bar{r}_0! \dots \bar{r}_m!} \right)^2 \\ &= \sum_{k=N_\gamma}^{\infty} K_\gamma (M_\gamma)^{k - N_\gamma} \frac{K^k}{k!} (m+1)^{2k} \\ &= K_\gamma (K(m+1)^2)^{N_\gamma} \sum_{k=0}^{\infty} \frac{1}{(k + N_\gamma)!} (M_\gamma)^k (K(m+1)^2)^k \\ &\leq K_\gamma (K(m+1)^2)^{N_\gamma} \sum_{k=0}^{\infty} \frac{1}{k!} M_\gamma^k (K(m+1)^2)^k = \\ &= K_\gamma ((m+1)^2 K)^{N_\gamma} e^{M_\gamma (m+1)^2 K} \end{aligned}$$

We have used the observation that for $\bar{r} = (\bar{r}_0, \dots, \bar{r}_m)$, with $|\bar{r}| = k$, $|\{w \in Q^* \mid r(w) = \bar{r}\}| = \frac{k!}{\bar{r}_0! \dots \bar{r}_m!}$.

In summary, Theorem 1 characterises the set $\mathcal{U}_{\gamma, T}$ of input trajectories U for which input output map of the original and reduced system are identical; whereas, Theorem 2 provides the error bounds for the inputs which do not belong to $\mathcal{U}_{\gamma, T}$.

4. MODEL REDUCTION BY NICE SELECTION

In this section, we present procedures for computing α -partial and β -partial realizations of an input-output map f which is realizable by a bilinear system.

Definition 3. Let Σ be a bilinear systems of the form (1). Let α be a nice row selection and β be a nice column selection related to Σ . Then the subspaces

$$\begin{aligned} \mathcal{O}_\alpha(\Sigma) &= \bigcap_{w \in \alpha} \ker CA_w \text{ and} \\ \mathcal{R}_\beta(\Sigma) &= \text{Span}\{A_w x_0 \mid w \in \beta\} \end{aligned}$$

will be called α -unobservability and β -reachability spaces of Σ respectively.

The spaces $\mathcal{O}_\alpha(\Sigma)$ and $\mathcal{R}_\beta(\Sigma)$ will be denoted by \mathcal{O}_α and \mathcal{R}_β , if Σ is clear from the context.

Theorem 3. Let $\Sigma = (p, m, n, \{A_i\}_{i \in Q}, C, x_0)$ be a bilinear system, and β be a nice column selection. Let $V \in \mathbb{R}^{n \times r}$ be a full column rank matrix such that

$$\mathcal{R}_\beta = \text{Im}(V),$$

and let V^{-1} be any left inverse of V . Define

$$\forall q \in Q : \bar{A}_q = V^{-1} A_q V, \quad \bar{C} = CV, \quad \bar{x}_0 = V^{-1} x_0.$$

Then $\bar{\Sigma} = (p, m, r, \{\bar{A}_q\}_{q \in Q}, \bar{C}, \bar{x}_0)$ is a β -partial realization of $f = Y_{\Sigma, x_0}$. Furthermore, the β -reachability spaces of $\bar{\Sigma}$ and Σ are equal, $\mathcal{R}_\beta(\bar{\Sigma}) = \mathcal{R}_\beta(\Sigma)$.

The proof of Theorem 3 follows the idea of the proofs of Theorems 3 and 6 in Bastug et al. (2016) and is omitted. By duality, we can formulate moment matching by nice row selections, as in Theorems 4.

Theorem 4. Let $\Sigma = (p, m, n, \{A_i\}_{i \in Q}, C, x_0)$ be a bilinear system and let α be a nice row selection. Let $W \in \mathbb{R}^{r \times n}$ be a full row rank matrix such that

$$\mathcal{O}_\alpha = \ker(W),$$

and let W^{-1} be any right inverse of W .

$$\forall q \in Q : \bar{A}_q = W A_q W^{-1}, \quad \bar{C} = CW^{-1}, \text{ and } \bar{x}_0 = W x_0.$$

Then $\bar{\Sigma} = (p, m, r, \{\bar{A}_q\}_{q \in Q}, \bar{C}, \bar{x}_0)$ is an α -partial realization of $f = Y_{\Sigma, x_0}$. Furthermore, the α -unobservability spaces of $\bar{\Sigma}$ and Σ are equal, $\mathcal{O}_\alpha(\bar{\Sigma}) = \mathcal{O}_\alpha(\Sigma)$.

Remark 3. (Choosing the model order). Note that using nice selections allows us to choose the order of the reduced system. Indeed, assume that γ is a nice row or column selection, and assume that γ has \bar{n} elements. It is then easy to see that $\dim \mathcal{R}_\gamma \leq \bar{n}$. If $p = 1$, i.e. there is one output, then $n - \dim \mathcal{O}_\gamma \leq \bar{n}$. Moreover, in this case, generically, $\dim \mathcal{R}_\gamma = \bar{n}$ and $\dim \mathcal{O}_\gamma = n - \bar{n}$. This can be shown in a manner similar to the the discussion after (Bastug et al., 2016, Theorem 7). Hence, Theorem 3 – 4 yield a reduced order model of order at most \bar{n} . Similarly to the proof of (Bastug et al., 2016, Theorem 7), it can be shown that if Σ is a minimal system, then for any $r = 1, \dots, n$, there exist a nice row selection α and a nice column selection β such that α and β both have r elements and $\dim \mathcal{R}_\beta = r$, $\dim \mathcal{O}_\alpha = n - r$, and Theorem 3 – 4 yield a reduced order model of order r . In Section 5, we illustrate this in a numerical example. The discussion above can be extended to the case with multiple outputs, if the definition of a nice row selection is modified to include the choice of the output channel. This can be done along the lines of (Bastug et al., 2016, Definition 4).

If the nice selections at hand are finite and of small size, then computing matrix representations of the spaces \mathcal{R}_β and \mathcal{O}_α is trivial. However, nice row and column selections need not to be finite, or if they are finite, their cardinality

can be large. For example, the nice selections which satisfy the conditions of Theorem 1 are always infinite, and the nice selection corresponding to classical N -partial realization (see Remark 1) is finite but its cardinality is exponential in N . Hence, for these cases, the question arises how to compute matrices V and W used in Theorem 3 – 4.

We will start by presenting a special case, when $\alpha = \{v \in Q^* \mid |v| \leq N\}$ and $\beta = \{v \in Q^* \mid |v| \leq N\}$. Denote \mathcal{R}_β by \mathcal{R}_N and \mathcal{O}_α by \mathcal{O}_N . Denote by $\text{orth}(M)$ the orthogonal matrix V such that V is full column rank, $\text{Im}(V) = \text{Im}(M)$ and $V^T V = I$. Consider the algorithms Algorithm 1 – 2.

Algorithm 1 Calculate a matrix representation of \mathcal{R}_N ,

Inputs: $(\{A_q\}_{q \in Q}, x_0)$ and N

Outputs: $V \in \mathbb{R}^{n \times r}$, $\text{rank}(V) = r$, $\text{Im}(V) = \mathcal{R}_N$.

```

1:  $V_0 := U_0; U_0 := x_0$ 
2: for  $k = 1 \dots N$  do
3:    $V := \text{orth}([U_0, A_0 V, A_1 V, \dots, A_m V])$ 
4: end for
5: return  $V$ .
```

Algorithm 2 Calculate a matrix representation of \mathcal{O}_N

Inputs: $(C, \{A_q\}_{q \in Q})$ and N

Output: $W \in \mathbb{R}^{r \times n}$, $\text{rank}(W) = r$ and $\ker(W) = \mathcal{O}_N$.

```

1:  $V := U_0, U_0 := C^T$ .
2: for  $k = 1 \dots N$  do
3:    $V := \text{orth}([U_0, A_0^T V, A_1^T V, \dots, A_m^T V])$ 
4: end for
5: return  $W = V^T$ .
```

Lemma 2. With the notation above, Algorithm 1 returns a matrix V such that $\text{Im}(V) = \mathcal{R}_N$ and $V^T V = I_r$ and Algorithm 2 returns a matrix W such that $\ker(W) = \mathcal{O}_N$.

Notice that the computational complexity of Algorithm 1 and Algorithm 2 is polynomial in N and n , even though the spaces of \mathcal{R}_N (resp. \mathcal{O}_N) are generated by images (resp. kernels) of exponentially many matrices. In fact, Algorithm 1 – 2 stop after N iterations.

For more general nice row and column selections, we need more sophisticated algorithms. To this end, we recall the concept of non-deterministic finite state automaton and its language.

Definition 4. 1 A non-deterministic finite state automaton (NFA) is a tuple $\mathcal{A} = (S, Q, \{\rightarrow_q\}_{q \in Q}, F, s_0)$ such that

- (1) S is the finite state set,
- (2) $F \subseteq S$ is the set of accepting (final) states,
- (3) $\rightarrow_q \subseteq S \times S$ is the state transition relation labelled by q , and
- (4) $s_0 \in S$ is the initial state.

For every $w \in Q^*$, define \rightarrow_w inductively as follows: $\rightarrow_\epsilon = \{(s, s) \mid s \in S\}$ and $\rightarrow_{wq} = \{(s_1, s_2) \in S \times S \mid \exists s_3 \in S : (s_1, s_3) \in \rightarrow_w \text{ and } (s_3, s_2) \in \rightarrow_q\}$ for all $q \in Q$. We denote the fact $(s_1, s_2) \in \rightarrow_w$ by $s_1 \rightarrow_w s_2$. Define the language $L(\mathcal{A})$ accepted by \mathcal{A} as

$$L(\mathcal{A}) = \{w \in Q^* \mid \exists s \in F : s_0 \rightarrow_w s\}.$$

We say that \mathcal{A} is *co-reachable*, if from any state a final state can be reached, i.e., for any $s \in S$, there exists $w \in Q^*$ and $s_f \in F$ such that $s \rightarrow_w s_f$. It is well-known that if \mathcal{A}

accepts L , then we can always compute an NFA \mathcal{A}_{co-r} from \mathcal{A} such that \mathcal{A}_{co-r} accepts L and it is co-reachable.

Consider a nice row selection α and a nice column selection β , and assume that α and β are regular languages. The matrix representations of the spaces \mathcal{R}_β and \mathcal{O}_α can be computed by Algorithms 3 – 4.

Algorithm 3 Calculate a matrix representation of \mathcal{R}_β ,

Inputs: $(\{A_q\}_{q \in Q}, x_0)$ and $\hat{\mathcal{A}} = (S, \{\rightarrow_q\}_{q \in Q}, F, s_0)$ such that $L(\hat{\mathcal{A}}) = \beta$, $F = \{s_{f_1}, \dots, s_{f_k}\}$, $k \geq 1$ and $\hat{\mathcal{A}}$ is co-reachable.

Outputs: $\hat{V} \in \mathbb{R}^{n \times \hat{r}}$, $\text{rank}(\hat{V}) = \hat{r}$, $\text{Im}(\hat{V}) = \mathcal{R}_\beta$.

```

1:  $\forall s \in S \setminus \{s_0\} : V_s := 0, V_{s_0} := \text{orth}(x_0)$ .
2: repeat
3:    $\forall s \in S : V_s^{old} := V_s$ 
4:   for  $s \in S$  do
5:      $M_s := V_s$ 
6:     for  $q \in Q, s' \in S : s' \rightarrow_q s$  do
7:        $M_s := [M_s, A_q V_{s'}^{old}]$ 
8:     end for
9:      $V_s := \text{orth}(M_s)$ 
10:  end for
11: until  $\forall s \in S : \text{rank}(V_s) = \text{rank}(V_s^{old})$ 
12: return  $\hat{V} := \text{orth}([V_{s_{f_1}} \dots V_{s_{f_k}}])$ .
```

Algorithm 4 Calculate a matrix representation of \mathcal{O}_α ,

Inputs: $(\{A_q\}_{q \in Q}, C)$ and $\hat{\mathcal{A}} = (S, \{\rightarrow_q\}_{q \in Q}, F, s_0)$ such that $L(\hat{\mathcal{A}}) = \alpha$, $F = \{s_{f_1}, \dots, s_{f_k}\}$, $k \geq 1$ and $\hat{\mathcal{A}}$ is co-reachable.

Outputs: $\hat{W} \in \mathbb{R}^{\hat{r} \times n}$, $\text{rank}(\hat{W}) = \hat{r}$, $\ker(\hat{W}) = \mathcal{O}_\alpha$.

```

1:  $\forall s \in S \setminus F : W_s := 0, \forall s \in F : W_s^T := \text{orth}((H)^T)$ .
2: repeat
3:    $\forall s \in S : W_s^{old} := W_s$ 
4:   for  $s \in S$  do
5:      $M_s := W_s$ 
6:     for  $q \in Q, s' \in S : s \rightarrow_q s'$  do
7:        $M_s := [M_s^T, (W_{s'}^{old} A_q)^T]^T$ 
8:     end for
9:      $W_s^T := \text{orth}(M_s^T)$ 
10:  end for
11: until  $\forall s \in S : \text{rank}(W_s) = \text{rank}(W_s^{old})$ 
12: return  $\hat{W} := W_{s_0}$ .
```

Lemma 3. Suppose that α and β are regular languages, i.e., there is a co-reachable NFA $\hat{\mathcal{A}}$ which accepts β respectively α . Algorithm 3 returns a full column rank matrix \hat{V} such that $\text{Im}(\hat{V}) = \mathcal{R}_\beta$, and Algorithm 4 returns a full row rank matrix \hat{W} such that $\ker(\hat{W}) = \mathcal{O}_\alpha$.

Algorithms 1 – 2 are particular instances of Algorithm 3 – 4 when applied to the nice selections $\alpha = \beta = \{v \in Q^* \mid |v| \leq N\}$ with the choice of accepting automaton $\hat{\mathcal{A}} = (S, Q, \{\rightarrow_q\}_{q \in Q}, S, 0)$, where $S = \{0, \dots, N\}$, $\rightarrow_q = \{(i, i+1) \mid i = 0, \dots, N-1\}$ for all $q \in Q$.

As a final remark, Algorithms 3–4 are essentially the same as (Bastug et al., 2016, Algorithm 1 – 2) and Lemma 3 is a simple consequence of (Bastug et al., 2016, Lemma 1). In particular, as it was explained in Bastug et al. (2016), the

computational complexity of Algorithms 3–4 is polynomial in n , and the algorithms stop after at most $n|S|$ iterations, where $|S|$ is the number of states of the NDFA $\hat{\mathcal{A}}$. However, $n|S|$ is the upper bound on the number of iterations, for particular examples the actual number of iterations can be much smaller.

5. EXAMPLE

To illustrate the methods developed in this paper, we consider a number of numerical examples of systems.

We start with a small dimensional example with the aim of illustrating Theorem 1 and Lemma 1. Consider the bilinear system Σ of the form (1), where

$$A_0 = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -0.1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T, \quad x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

with $m = 3$, $n = 4$. Consider the following nice column selection γ accepted by an NDFA of the form $\mathcal{A} = (S, Q, \{\rightarrow_q\}_{q \in Q}, F, s_0)$, where $S = \{1, 2, 3\} = F$, $s_0 = 1$ and $\rightarrow_0 = \{(i, j) \mid i \leq j\} \cup \{(3, 1)\}$ and for all $q = 1, \dots, m$, $\rightarrow_q = \{(i, q) \mid i \leq q\}$. In other words, $\gamma = L(\mathcal{A})$. More explicitly, γ can be described as follows. Let L be the set of all the sequences $w \in Q^*$ such that $w = v_1 v_2 v_3$, $v_i \in \{0, i\}^*$, $i = 1, 2, 3$. Then γ is the set of words of the form $w_1 w_2 \dots w_k$, $k \geq 1$, such that $w_1 \in L$ and for all $i = 2, \dots, k$, $w_i = 0 w'_i$ where $w'_i \in L$. To illustrate, the word $1^* 2^* 3^* 0^* 2^* 3^* \in \gamma$, whereas $2^{l_1} 1^{l_2} 3^*$ for $l_1, l_2 \in \mathbb{N} \setminus \{0\}$ does not belong to γ .

Algorithm 3 and Theorem 3 yield a reduced order model $\bar{\Sigma} = (\bar{C}, \{\bar{A}\}_{q \in Q}, \bar{x}_0)$ of size $r = 3$, where

$$\bar{A}_0 = \bar{A}_1 = \mathbf{O}_{3 \times 3}, \quad \bar{A}_1 = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -3 & 0 & -0.1 \end{bmatrix}$$

$$\bar{C} = [1 \ 0 \ 0], \quad \bar{x}_0 = [0 \ 1 \ 0]^T$$

and $\mathbf{O}_{3 \times 3}$ denotes the 3×3 zero matrix. By Theorem 3, $\bar{\Sigma}$ is a γ -partial realization of the input-output map $f = Y_{\Sigma, x_0}$; and by Theorem 1, for any $u \in \mathcal{U}_{\gamma, T}$, the outputs of $\bar{\Sigma}$ and Σ are equal on $[0, T]$. For example, consider the input

$$u(t) = \begin{cases} (\cos(t\pi) + 2, 0, 0)^T & t \in [0, 0.1) \\ (0, \cos(t\pi) + 2, 0)^T & t \in [0.1, 0.2) \\ (0, 0, \cos(t\pi) + 2)^T & t \in [0.2, 5) \\ (0, 0, 0)^T & t \in [5, 6) \\ (0, \cos(t\pi) + 2, 0)^T & t \in [6.1, 6.2) \\ (0, 0, \cos(t\pi) + 2)^T & t \in [6.1, +\infty) \end{cases} \quad (11)$$

then by Theorem 1, $u \in \mathcal{U}_{\gamma, T}$ for all $T \geq 6.1$ and hence $y = Y_{\Sigma, x_0}(u)$ and $\bar{y} = Y_{\bar{\Sigma}, \bar{x}_0}(u)$ coincide on $[0, T]$. The corresponding response is shown on Figure 1 for $T = 10$. Although according to Theorem 1, these two responses should be equal, however, on Figure 1 one sees a slight difference, which is due to numerical error.

However, for u given by

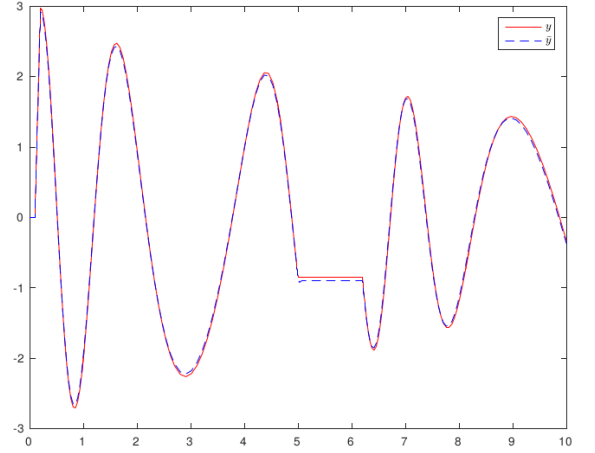


Fig. 1. Comparisons of responses of Σ (red) and $\bar{\Sigma}$ (green) for u defined in (11).

$$u(t) = \begin{cases} (0, \sin(t\pi) + 2, 0)^T & t \in [0, 0.5) \\ (\sin(t\pi) + 2, 0, 0)^T & t \in [0.5, 1) \\ (0, 0, \sin(t\pi) + 2)^T & t \in [1, +\infty) \end{cases} \quad (12)$$

$u \notin \mathcal{U}_{\gamma, T}$, $T = 10$. The corresponding outputs $y = Y_{\Sigma, x_0}(u)$ and $\bar{y} = Y_{\bar{\Sigma}, \bar{x}_0}(u)$ need not coincide on $[0, T]$. The corresponding response are shown in Figure 2 for $T = 10$. In fact, Σ is exponentially unstable for this u , while the state and output trajectory of $\bar{\Sigma}$ remain bounded.

Note that if we choose the nice selection $\alpha = \{\epsilon, 3, 34\}$, and we apply Theorem 3 to Σ above, then \mathcal{R}_α will be an invertible 3×3 matrix and in this case Theorem 3 yields a reduced order bilinear system which is related to $\bar{\Sigma}$ described above by a linear isomorphism. If we apply Theorem 3 to $\alpha = \{\epsilon, 3\}$, then we obtain a reduced order model of order 2. This illustrates that finite nice selections can be used to choose the order of the reduced model, as explained in Remark 3. The code for this example

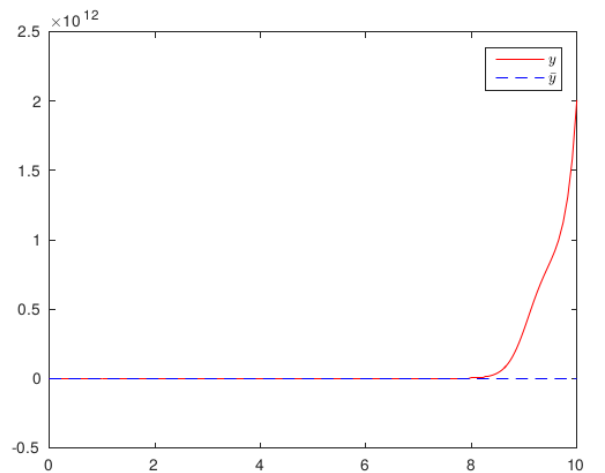


Fig. 2. Comparisons of responses of Σ (red) and $\bar{\Sigma}$ (blue) for u defined in (12).

can be found in the supplementary material, in the file `BilinearModelExample2.m`.

To demonstrate the scalability of the proposed approach, we tested it on a bilinear system Σ of the form (1) with $m = 4$ inputs and $n = 200$ states and one output. In this case, γ was chosen to be the language accepted by an NDFA of the form $\mathcal{A} = (S, Q, \{\rightarrow_q\}_{q \in Q}, F, s_0)$, where $S = \{1, 2, \dots, m\} = F$, $s_0 = 1$ and $\rightarrow_0 = \{(i, j) \mid i \leq j\}$ and for all $q = 1, \dots, m$, $\rightarrow_q = \{(i, q) \mid i \leq q\}$. Hence, $L(\mathcal{A}) = \gamma$. More explicitly, γ can be described as the set of all the sequences $w \in Q^*$ such that $w = v_1 \dots v_m$, $v_i \in \{0, i\}^*$, $i = 1, \dots, m$. Applying Algorithm 3 and Theorem 3 to Σ yielded a bilinear system $\bar{\Sigma} = (1, 4, r, \{\bar{A}_q\}_{q \in Q}, \bar{C}, \bar{x}_0)$ of order $r = 9$. In this case, it took 4 iterations for Algorithm 3 to terminate, which is much smaller than the theoretical upper bound $200 \cdot 4$. Due to lack of space, we do not present the matrices of Σ and $\bar{\Sigma}$, the tex files with the matrices and the .mat files can be found among the supplementary material of this report. For each $i = 0, \dots, m$, the matrices A_i are in the files with .tex and .mat extension called `NOLCOSBig_exampeAi`. The matrix s C is stored in the .tex and .mat file `NOLCOSBig_exampeC`, and the initial state x_0 is in the file `NOLCOSBig_exampeX0`. In a similar manner, for each $i = 0, \dots, m$, A_i is stored in files called `NOLCOSBig_exampeAi_r`, and \bar{C} and \bar{x}_0 are stored in the files named `NOLCOSBig_exampeC_r`, and `NOLCOSBig_exampeX0_r`. By Theorem 1, $\bar{\Sigma}$ has the same output as Σ on $[0, T]$, $T = 50$, where for the input u which satisfies $u(t) = \cos(\pi t)e_i$ if $t \in [t_{i-1}, t_i)$ for all $i = 1, \dots, m$, where $t_0 = 0$ and $t_{i+1} = t_i + 10$, $i = 0, \dots, m - 1$. This complies with Theorem 1, as $u \in \mathcal{U}_{\gamma, T}$. The responses $y = Y_{\Sigma, x_0}(u)$ and $\bar{y} = Y_{\bar{\Sigma}, \bar{x}_0}(u)$ are shown on Figure 3, and it can be seen that they are indeed the same.

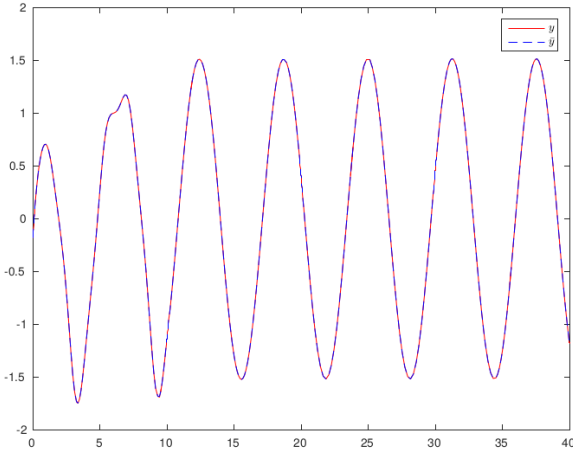


Fig. 3. Comparisons of responses of Σ (red) and $\bar{\Sigma}$ (green) for $n = 200$, $r = 200$, $m = 4$.

We also applied Algorithm 1 and Theorem 3 to Σ for $\gamma = \{v \in Q^* \mid |v| \leq N\}$ with $N = 3$. In this case, we obtained a reduced-order system $\hat{\Sigma} = (1, 4, n_r, \{\hat{A}_q\}_{q \in Q}, \hat{C}, \hat{x}_0)$ of order $n_r = 17$. The matrices \hat{A}_q , $q \in Q$, \hat{C} and the vector \hat{x}_0 can be found in the supplementary material, in the files named `NOLCOSBig_exampeAqpr`, `NOLCOSBig_exampeCpr`, `NOLCOSBig_exampeX0pr` respectively. We evaluated the response y and \hat{y} of Σ and $\hat{\Sigma}$ respectively for the following input: $u(t) = (1 + \cos(\pi t))e_{q_i} + (\cos(t), \cos(2t), \dots, \cos(mD))^T$ if $t \in [t_{i-1}, t_i)$, $i =$

$1, \dots, k$, where $k = 5$, $q_1 = 4$, $q_2 = 3$, $q_3 = 1$, $q_4 = 2$, $q_5 = 4$, $t_0 = 0$ and $t_{i+1} = t_i + 10$, $i = 0, \dots, k - 1$, see Figure 4. We observe that y and \hat{y} are not the same, but on $[0, T]$, $T = 10$ they are close as predicted by Theorem 2. The

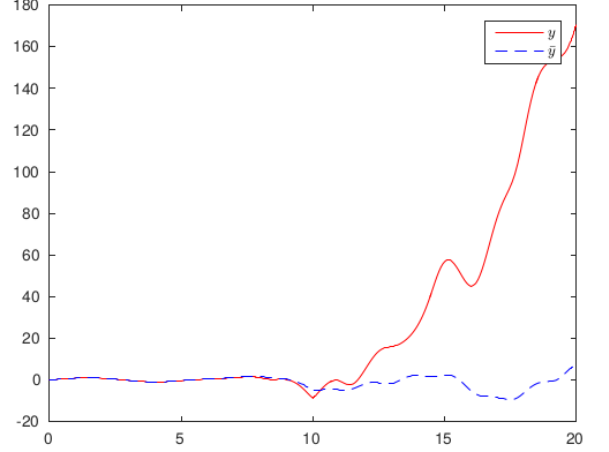


Fig. 4. Comparisons of responses of Σ (red) and $\bar{\Sigma}$ (green) for $n = 200$, $r = 200$, $m = 4$.

code for this example can be found in the supplementary material, in the file `BilinearModelExample1.m`.

Finally, we evaluated the proposed method on the nonlinear circuit investigated in Breiten and Damm (2010); Bai and Skoogh (2006) with $N_o = 150$ nonlinear resistors. Following these references, we applied Carleman bilinearization to the original model. As the results, the obtained bilinear system $\Sigma = (1, 2, n, \{A_i\}_{i=0}^1, C, x_0)$ is of the order $n = N_o + N_o^2 + 1$. The code for generating the model can be found in the supplementary material, in the file `NonlinearCircuitBilinModel.m`. In order to obtain matrices C, A_0, A_1, x_0 , the code in `NonlinearCircuitBilinModel.m` should be run. After the code has been run, the matrices C, A_0, A_1, x_0 will be stored in the files `CNonlinRC.mat`, `ANonlinRC.mat`, `NNonlinRC.mat`, `x0NonlinRC.mat` respectively.

We applied Algorithm 4 and Theorem 4 with $\alpha = \{\epsilon, 0, 1, 10, 110, 1110, 1110\}$, the resulting reduced order bilinear system $\bar{\Sigma} = (1, 2, 6, \{\bar{A}_i\}_{i=0}^1, \bar{C}, \bar{x}_0)$ (of order 6). The matrices $\bar{C}, \bar{A}_0, \bar{A}_1, \bar{x}_0$ can be found in the supplementary material, in the files `CNonlinRCnr.mat`, `ANonlinRCnr.mat`, `NNonlinRCnr.mat`, `x0NonlinRCnr.mat` respectively. Algorithm 4 took 5 iterations to stop, which is much less than the theoretical upper bound of $N_o|S|$, where $|S|$ is the number of states of the NDFA accepting α . Note that $N_o = 150$ and $|S| = 7$ in this case. We applied Theorem 4 and Algorithm 2 with $N = 3$, which yielded a reduced order system $\hat{\Sigma} = (1, 2, 12, \{\hat{A}_i\}_{i=0}^1, \hat{C}, \hat{x}_0)$ (of order 12). The matrices $\hat{C}, \hat{A}_0, \hat{A}_1, \hat{x}_0$ can be found in the supplementary material, in the files `CNonlinRCr.mat`, `ANonlinRCr.mat`, `NNonlinRCr.mat`, `x0NonlinRCr.mat` respectively.

We simulated the responses of the original nonlinear model (before Carleman's bilinearization), and the responses $\bar{y} = Y_{\bar{\Sigma}, \bar{x}_0}$, $\hat{y} = Y_{\hat{\Sigma}, \hat{x}_0}$ of $\bar{\Sigma}$ and $\hat{\Sigma}$ for the input $u(t) = (\cos(2\pi t/10) + 1)/2$, the result is shown on Figure 5. The responses of $\bar{\Sigma}$ and $\hat{\Sigma}$ are both reasonably close to the response of the original nonlinear model, yet the order of $\bar{\Sigma}$ is much lower than that of $\hat{\Sigma}$. This demonstrates that nice selections give additional flexibility to model reduction. The code for this example can be found in the supplemen-

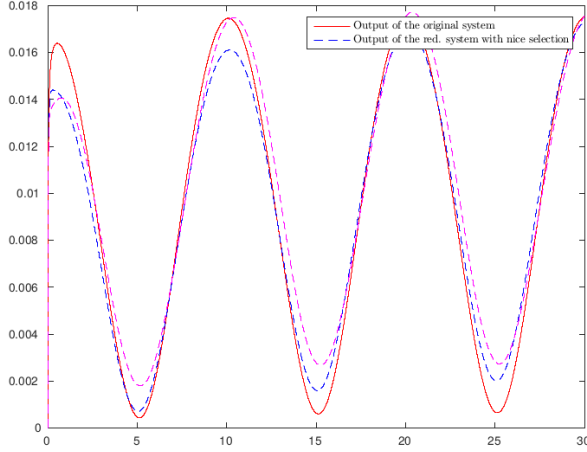


Fig. 5. Comparisons of responses of the nonlinear RC and \bar{y} and \hat{y} of $\bar{\Sigma}$ and $\hat{\Sigma}$

tary material, in the file `TestNPartialNonlinCircuit.m`.

6. CONCLUSION

We have developed a method for model reduction of bilinear control systems leaning upon the concept of the column nice selection and the row nice selection. The resulting bilinear system has exactly the same output response as the original system for inputs consistent with a nice selection. For other inputs, the error between the time responses of the two systems decreases with the cardinality of the nice selection, provided the inputs and considered time horizon are short. Furthermore, we have provided algorithms for computing matrix representations of α -unobservability and β -reachability spaces, which has been used for computing α -partial and β -partial realizations of an input-output map. Future research will be directed towards a better understanding of the numerical issues involved, of error bounds for the reduced model, and of the relative advantage of the proposed method in comparison to Bai and Skoogh (2006); Feng and Benner (2007); Lin et al. (2007); Flagg (2012); Breiten and Damm (2010); Benner and Breiten (2015); Flagg and Gugercin (2015); Wang and Jiang (2012).

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